Understanding mathematical models describing vibration based induction generators for energy harvesting

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**Originality Statement**

I have worked in conjunction with Lamar University faculty to produce a unique paper which addresses current research in the field of energy harvesting. I have used information from multiple sources, but to the best of my knowledge, the ideas and conclusions presented in this paper are my own. The information used to build these ideas is not unique, but its application in this paper and my specific conclusions are my own.

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Abstract

The study and implementation of energy harvesting technology is expanding rapidly. Vibration based induction generators are receiving major attention in areas of research and industry. Understanding the mathematical and physical properties behind induction harvesters is necessary for designing and implementation into current technology. Two examples of induction generator mathematical models are presented and discussed. Each is broken down into simple terms to facilitate understanding of the systems and their importance. This knowledge can then be used to design better systems and advance research in this topic.

Introduction

Energy harvesting dates back to the use of windmills and water wheels. We have been searching since then to find better ways of collecting and storing energy found all around us. Recently we have seen large wind farms, solar collector systems, hydroelectric, and geothermal systems bring large scale energy production to the national grids. The current research is pushing these technologies into small scale applications, and due to the continued advances in low power electronics they are becoming viable.

These small scale harvesting devices gather power from multiple sources, each with different advantages and tradeoffs. Solar power is very energy dense for its size, but it is only effective in environments with lots of ambient light. Wind power is available but only for certain stationary applications like wireless sensors. Thermal couplers can use temperature gradients to generate power, but the energy density is related to the size of temperature gradient. In small scale energy harvesting the gradients is very small which translates into very low power densities. Piezoelectric devices generate power by deforming a cantilever beam using vibration sources. They are effective, but only in situations where ambient vibration is available. Electromagnetic generators produce power by changing the electromagnetic flux passing through a conducting coil. Understanding the advantages and tradeoffs of these generation sources will spur future development of these alternative power sources [A].

The electromagnetic induction generator is a device widely used in multiple power applications today. Large scale power plants all the way down to tiny wind generators use a form of induction generator to harvest rotational energy and transform it into electrical energy. For many years studies have been preformed to improve and modernize these types of generators. The losses in large scale inductions motors are well known and research has been done to reduce these losses thus providing more power for our everyday use.

An induction generator in its simplest form is a coil of conducting material whose terminals are placed across a load. A changing magnetic flux in induced through the coils. This changing flux induces a current in the coil which builds and equal magnetic flux but in the opposite direction. The current induced in the coil is used to drive the circuit [C].

More recently, induction generators have become a topic of discussion in the area of energy harvesting. Many scholars believe that induction generators might be very useful as vibration energy harvester. Studies have been conducted to determine the possible power production an induction generator could sustain and further work has gone into the design of some small scale generators. These induction harvesters are very small and are expected to produce very low levels of power. Our current means of investigation of the induction generator is through sophisticated mathematical modeling.

The goal of this paper is to provide a simple background to the process of modeling induction generators, and present examples of current induction generator models. This background will be useful for engineers designing devices which could include vibration based induction generators in their power schemes.
The theory behind induction harvesters is well known and understood. The mathematical formulas used to explain these generators were developed over one hundred years ago by Gauss, Maxwell, Faraday, and Ampere [C]. These equations form the backbone of all mathematical models and are able to give their users a wealth of information before construction begins.

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = 4\pi k \rho \quad \int \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \\
\nabla \cdot \vec{B} = 0 \quad \int \vec{B} \cdot d\vec{A} = 0 \\
\n\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \int \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \\
\n\nabla \times \vec{B} = \frac{\vec{J}}{\varepsilon_0 c^2} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad \int \vec{B} \cdot d\vec{s} = \mu_0 i + \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A}
\n\]

(Figure 1: Maxwell’s Equations [B])

Researchers use these equations to build working mathematical models of physical systems. With these models they can then look at ways to meet needs described by individual designs. The power behind the model is the ability to take an abstract idea and convert it into an easily understood and usable system. With this system it is then possible to create an electronic representation of multiple prototypes to be tested. If any of these prototypes meet the given specifications and are feasible, then they can move from the computer to a physical device for testing. Using the mathematical model takes out unnecessary construction and physical testing thus making the design process simpler and more efficient overall.

Using physics, the basic formulas are broken down and their components are defined using physical phenomena described by the equations and inherent in induction generators. This process makes the equations more complex, but also more useful. Components such as flux or current and better understood when explained using permanents magnets of specific area and strength, or voltage difference per ohm. Adding these descriptive equivalent components helps us understand exactly how the induction harvester will perform.

Using the variables from these derived equations, researchers can adjust the design of individual harvesters based on any number of needs. For example, if one knew the energy requirements of the circuit, and the maximum dimensions of the device, this information could be used to determine the dimensions and characteristics of the generator required by the device [D].

Determining just how these basic equations should be modified is the first step in creating a useful model.

Shad Roundy put forth a way to model vibration based electromagnetic induction harvesters and their efficiency using the following mathematics [E].

\[
\begin{bmatrix}
F \\
\lambda'
\end{bmatrix} = \begin{bmatrix}
k_{sp} & Bl \\
Bl & L
\end{bmatrix} \begin{bmatrix}
i \\
l
\end{bmatrix}
\]

(Figure 1: Maxwell’s Equations [B])

In this equation \( F \) is the force applied to the proof mass, \( \lambda ' \) is the flux linkage between magnetic and coil, \( k_{sp} \) is the stiffness of the restoring spring connected to the proof mass, \( B \) is the magnetic field, \( l \) is the total length of the conductor (coil), \( L \) is
the inductance of the conductor (coil), $z$ is the relative displacement of the conductor and magnetic field, and $i$ is the current in the conductor. Roundy then solves for the coupling coefficient $k$ using the following equation.

$$k^2 = \frac{(BI)^2}{k_{sp}L} \quad [2]$$

Next Roundy solves for the maximum output energy.

$$U_{max} = \frac{(BI)^2}{4L} z^2 = \frac{k_{sp}k^2}{4} z^2 \quad [3]$$

The input energy is then determined.

$$U_{in} = \left(k_{sp} - \frac{(BI)^2}{2L}\right) Z^2 = \left(1 - \frac{k^2}{2}\right) k_{sp} Z^2 \quad [4]$$

The maximum transmission coefficient in terms of $z$ can then be found.

$$\lambda_{max} = \frac{k^2}{4 - 2k^2} = \frac{1}{2} \frac{(BI)^2}{2k_{sp}L - (BI)^2} \quad [5]$$

Roundy expands his model to utilize vibration energy by describing its use to find the displacement $z$ between the conductor and the proof mass magnetic.

$$z(t) = Qy(t), \quad \text{where } Q = \frac{1}{(2\zeta)} \quad [6]$$

In these equations $Q$ is the quality factor and $\zeta$ is the dimensionless damping ratio and is related to the dampening constant $b$ established by the system.

$$b = 2m\zeta \omega_n, \quad \text{where } \omega_n \text{ is the natural frequency} \quad [7]$$

The magnitude of the deflection is then described using these equations.

$$|Z| = Q|Y|, \text{ and } |A| = \omega^2|Y|, \text{ thus } |Z| = \frac{QA}{\omega^2} \quad [8]$$

Given that $A$ is the acceleration magnitude of the input vibrations. Now that the displacement $z$ is known, it can be substituted into the energy input equation and be used to find the power density of the system.

$$U_{in} = \left(1 - \frac{k^2}{2}\right) k_{sp} \frac{(QA)^2}{\omega^2}, \quad \text{where } k_{sp} = m\omega_n^2, \quad \text{and } \omega = \omega_n \quad [9]$$

$$U_{in} = \left(1 - \frac{k^2}{2}\right) \frac{m(QA)^2}{\omega^2} \quad [10]$$

$$p_{max} = \frac{k^2 m(QA)^2}{4\omega} = \frac{k^2 \rho(QA)^2}{4w}, \quad \rho \text{ is the density of the proof mass material} \quad [11]$$

Roundy’s model may seem very complex, but it is derived from very basic electromagnetic and dynamic system principles. These formulas can be used to describe any physical system utilizing a simple one axis vibration to create a displacement between a magnet and a conducting coil. If a certain number of variables were known, the results of the design could then be
extrapolated. Any system parameter could be solved for if the necessary conditions were first given or estimated. This particular model focuses on the power density as the goal design parameter. If the power requirements of the circuit are known, then the other designing parameters can be fitted together to create this power output through measurement or estimation. Multiple iterations could be used to design a generator that would fit a particular application using this model.

(Figure 2: A simple example of a possible system as described by the previous equations [E])

This model would be useful for designing a device which would rest on a vibrating piece of equipment. This type of harvester would be exposed to a set frequency of certain amplitude. Its output energy could then be set by determining the size of the harvester, density of the magnetic, strength of the magnetic, stiffness of the spring support, number of coils, cross sectional area of the coils, and so forth. Changing any or all of these will adjust the output energy and this flexibility gives designers the ability to input constraints and verify possible solutions in the design process. Models like this one would also help quickly resolve the question, “Could it work under these conditions?” [F].

Another model has been described by S P Beeby et al. in the paper Energy Harvesting Vibration Sources for Microsystems Applications. Their mathematical model takes a different approach in describing the same system shown in the previous model [G].

Beeby et al. create a model for inertial-based generators using a second-order spring-mass system. In this model $m$ is the seismic mass, $k$ is the stiffness of the spring, $c_p$ is the parasitic losses, $c_e$ is the losses due to the electrical energy extracted by the transduction mechanism. The two losses ($c_p$ and $c_e$) are expressed in the form of a damping coefficient $c_T$. The system is excited by an external sinusoidal vibration in the form

$$y(t) = Ysin(\omega t) \quad [12]$$
The movement caused by the outside excitation causes a net motion of the proof mass coil system of displacement \( z(t) \). Assuming the excitation is harmonic in nature, and that the mass causing the excitation is much larger than the proof mass, the motion of the system can be described using the following differential equation.

\[
m\ddot{z}(t) + c\dot{z}(t) + kz(t) = -m\ddot{y}(t) \quad [13]
\]

The standard steady state solution for the mass displacement equation is given as shown below with \( \omega \) as the frequency.

\[
z(t) = \frac{\omega^2}{\sqrt{(k/m - \omega^2)^2 + (cT\omega/m)^2}} Y\sin(\omega t - \phi) \quad [14]
\]

The phase angle \( \phi \) is described in the following equation.

\[
\phi = \tan^{-1}\left(\frac{cT\omega}{(k - \omega^2m)}\right) \quad [15]
\]

Maximum energy is obtained when the frequency used is the natural frequency of the system.

\[
\omega_n = \sqrt\frac{k}{m} \quad [16]
\]

The power lost to the transduction mechanism and the parasitic losses can be calculated using the formula below.

\[
P_d = \frac{m\zeta_T Y^2 \left(\frac{\omega}{\omega_n}\right)^3 \omega^3}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta_T \left(\frac{\omega}{\omega_n}\right)\right)^2}, \quad \text{when } \zeta_T = \frac{cT}{2m\omega_n} \quad [17]
\]

Maximum power generated by the device occurs when it is operated at the natural frequency \( \omega_n \). When \( \omega \) is equal to \( \omega_n \) the previous equation simplifies to the one shown below.

\[
P_d = \frac{mY^2\omega_n^2}{4\zeta_T} \quad [18]
\]

The variable \( A \) is the excitation acceleration level and is derived using the following formula.

\[
A = \omega_n^2 Y \quad [19]
\]

These equations are generated from a steady state solution. This means that the derived values can not rise up past a certain limit. The damping coefficients keep the system in the boundaries of reality. This model makes it possible to determine the power extracted from the system by the transducer by substituting the damping coefficients into the maximum power equation.

\[
P_e = \frac{m\zeta_e A^2}{4\omega_n \left(\zeta_p + \zeta_e\right)^2} \quad [20]
\]

The power extracted by the transducer is maximized when \( \ddot{x}_p = \ddot{x}_e \).

The power output of this device is inversely proportional to the natural frequency of the generator for a given excitation acceleration. This property encourages us to use the lowest possible fundamental frequency.
\[ c_e = \frac{(NIB)^2}{R_{load} + R_{coil} + j\omega L_{coil}} \] [21]

In this equation \( N \) is the number of turns in the coil, \( R \) is the resistance produced either by the load or the coil, and \( L \) is the inductive resistance (reactance) of the coil. This equation suggests that the load resistance \( R_{load} \) can be adjusted to match the two damping coefficients. By adjusting the value of \( R_{load} \), we can move the value of \( c_e \) closer to the value of \( c_p \) assuming this adjustment is allowed by the design parameters.

The best value for \( R_{load} \) can be found with this equation.

\[ R_{load} = R_{coil} + \frac{(NIB)^2}{c_p} \] [22]

With this value the maximum average power can be determined.

\[ P_{e_{load \ max}} = \frac{mA^2}{16\delta_p \omega_n} \left(1 - \frac{R_{coil}}{R_{load}}\right) \] [23]

These equations effectively describe the model put forth by Beeby and they describe the actions and responses of many common vibration based induction generators. It could be used to describe or design a one axis vibration based induction generator. The model presented by Beeby focuses on the damping coefficients and their contributions. It encourages matching to maximize output and put a larger emphasis on this approach.

(Figure 3: A simple example of a possible system as described by the previous equations [Gi])

These models are just two of many different models designed for this particular type of induction generator. These particular models were designed for the purpose of testing generators being developed in research departments and generators being produced and sold in the industry. The models were used to measure and compare performance of different devices, and to determine what situations where they might be useful.
Induction generator models are also necessary for designing efficient energy harvesters. Devices which might utilize and energy harvester in its power structure will need to use a model to find the right configuration for its particular needs. If a certain amount of power is needed and certain limitations are place on overall size; these design parameters can be placed in the model to determine if a generator would be possible. Once the model describes the possible generator the developers can determine if the model’s proposed generator is feasible. From here more parameters could be adjusted to improve certain aspects of the generator until a suitable harvesting device is designed. Models provide a way to perform preliminary design and testing of these devices without needing to build a prototype. They could also be used to test harvester available on the market to determine their potential use in specific application designs.

Conclusion

Energy harvesting is an expanding field of research and development. Vibration based induction generators will play a significant part in the development and implementation of energy harvesting technology. Understanding the physics and mathematics behind induction generators will be critical in developing usable devices. The easiest way to build this understanding is with a mathematical model that describes the generators physical properties. These theoretical models along with many others available provide a sound base upon which engineers can design and experiment. The research presented in each of these models will continue to fuel further development in this discipline. This development will lead to better devices on the market and greater innovation in the industry.

Further effort should be made to understand the physical properties of induction generators and more research is needed in accurately modeling specific orientations and configurations available for vibration based induction generation. Advancements in these areas are the next steps to improving our understanding and implementation of this technology.

References


