The idea of windowing

Recall: \[ X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n} \] - not really practical (6.2.1)

What if we observe an infinitely long sequence over a finite length time window?

Then we don’t see the rest of the signal.

Let \( D \) be the number of samples (data points) observed.
Windowing; what it causes

We specify \( y_n = W_n^R x_n \) - a windowed view of \( x_n \)

\[
W_n^R = \begin{cases} 
1, & n = 0, 1, \ldots, D-1 \\
0, & \text{otherwise}
\end{cases}
\]

\( y_n \) is a D-sequence (finite length), therefore, we can evaluate its DFT.

\[
y_n \leftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_n^R(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta
\]

periodic convolution property

Therefore, \( Y(e^{j\omega}) \) is a "dispersed/convoluted/obscured view of \( X(e^{j\omega}) \)

\[
y_n \leftrightarrow Y_k = Y(e^{j\omega})|_{\omega = \frac{2\pi}{P} k}, \quad k = 0, 1, \ldots, P-1
\]

Windowing; what it causes (cont)

Let \( x_n = e^{j\omega_n} \) - a sinusoid

\[
Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_n^R(e^{j\theta}) 2\pi \sum_l \delta(\omega - \omega_l + 2\pi l) d\theta = W_n^R(e^{jD\omega_l})
\]

if \( \omega_l \in [-\pi, \pi] \)

Frequency sampling \( \rightarrow Y_k = W_n^R \left( e^{j\frac{2\pi k}{P}} \right) \), \( k = 0, 1, \ldots, P-1 \)

delay due to a center of the window

\[
W_n^R(e^{j\omega}) = \sum_{k=0}^{D-1} W_n^R e^{-j\omega_n} = e^{-jD\omega} \left( e^{j\frac{D}{2}} - e^{-j\frac{D}{2}} \right)
\]

\[
= e^{-jD\omega} \frac{\sin(D\omega)}{\sin(\omega)}
\]
Leakage

\[ W^k(e^{j\omega}) = 0 \Rightarrow \omega_k = \frac{2\pi}{D} l, \quad l = 1, 2, \ldots \]  

(6.5.1)

When \( \omega \to 0 \); \( \frac{\sin(\omega D / 2)}{\sin(\omega / 2)} \to D \)

Let \( D = 100 \)

Spectrum of a rectangular window

This is what causes so called "leakage" – observation of frequency components that do not exist in the spectrum of the signal – Due to observation only!

Frequency sampling

DFT means "frequency sampling"!

We can only observe specific frequency components of the sinc curve, and, therefore, can see only the specific frequency components of our signal!

Zero-padding increases number of observed frequency samples.

BUT! our sinusoid leaves at \( \omega_0 \) and this is the only frequency where we expect it to be! Therefore, by "smart" choice of \( P \) we can observe the signal ONLY at \( \omega_0 \) and at the zero-crossings, i.e. at

\[ \omega = \omega_0 \pm \frac{2\pi}{P}, \quad l = 0, 1, \ldots \]  

(6.5.1)
Characteristics of window functions

Windows characteristics

We want:

- MLW – narrow for better spectral resolution
- PSL – lower to have less masking for nearby components
- SLR – “better” (faster) to have less masking for far away components
Multiple sinusoids

Usually: \( x_n = \sum_m A_m e^{i(\omega_m n + \varphi_m)} \) — an input is a mixture of sinusoids \( (6.9.1) \)

Remember: \( \cos(\omega_m n + \varphi_m) = \frac{1}{2} e^{i\omega_m n} + \frac{1}{2} e^{-i\omega_m n} \) \( (6.9.2) \)

Assume for simplicity that \( \varphi_m = 0 \) for every \( m \).

\[
X(e^{i\omega}) = \sum_m A_m 2\pi \sum_j \delta(\omega - \omega_m + 2\pi j) \quad (6.9.3)
\]

\[
Y(e^{i\omega}) = \sum_m A_m W^R \left( e^{i(\omega-\omega_m)} \right) \quad \Rightarrow \quad Y_j = \sum_m A_m W^R \left( e^{i\left(\frac{2\pi j}{D}\right)} \right) \quad (6.9.4)
\]

As a consequence, for two equal amplitude sinusoids, we will observe 2 peaks of \( |Y_k| \) "about" half the time (depending on relative phase) when the frequency difference is:

\[
\Delta \omega = \omega_1 - \omega_2 = \frac{2\pi}{D} \quad - \text{Rayleigh limit of frequency resolution} \quad - \text{related to the window!} \quad (6.9.5)
\]
Multiple sinusoids

Phase shift 180°

Two sinusoids are NOT resolved! They appear as a single peak in the frequency domain!

DFTs of two sinusoids of different frequencies and different magnitudes.  
DFT of their sum…
Multiple sinusoids

Masking – one signal is obscured by another in the freq. domain

We need a “better” window!

\[ s_1 = \sin\left(\frac{n}{3}\right) \]
\[ s_2 = 0.1\sin\left(\frac{n}{3} + 5\pi/D\right) \]

Multiple sinusoids

Alternative – von Hann window

Two sins can be resolved!!
Multiple sinusoids

Two windows comparison

Magnitude (dB)

Normalized Frequency (×π rad/sample)

Different DFT windows

Rectangular (Boxcar) window

\[ W_n^R = 1, \quad n = 0, 1, \ldots, D - 1 \] (6.14.1)

Window function (rectangular)

Spectral "leakage" from a sinusoid
Different DFT windows

von Hann (Hanning) window

\[ W_n^N = \frac{1}{2} \left(1 - \cos \left(\frac{2\pi n}{D-1}\right)\right) \approx \sin^2 \left(\frac{2\pi n}{D-1}\right), \quad n = 0, 1, \ldots, D-1 \]  \hspace{1cm} (6.15.1)

Hamming window

\[ W_n^M = 0.54 - 0.46 \cos \left(\frac{2\pi n}{D-1}\right), \quad n = 0, 1, \ldots, D-1 \]  \hspace{1cm} (6.16.1)
Different DFT windows

Blackman window

\[ W_n^B = 0.42 - 0.5 \cos \left( \frac{2\pi n}{D-1} \right) + 0.08 \cos \left( \frac{4\pi n}{D-1} \right), \quad n = 0,1,...,D-1 \]  \hspace{1cm} (6.17.1)

Different DFT windows

Kaiser window

\[ W_n^K = I_0 \left( \frac{\pi \alpha}{I_0 \left( \frac{\pi \alpha}{\sqrt{1 - \left( \frac{2n}{D-1} \right)^2} \right)} \right), \quad n = 0,1,...,D-1 \]  \hspace{1cm} (6.18.1)
## Different DFT windows

### Different windows comparison

<table>
<thead>
<tr>
<th>Window</th>
<th>MLW (2π/D)</th>
<th>PSL (dB)</th>
<th>SLR (dB/oct)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$\approx 0.9$</td>
<td>-13</td>
<td>-6</td>
</tr>
<tr>
<td>Hanning</td>
<td>$\approx 2$</td>
<td>-31</td>
<td>-18</td>
</tr>
<tr>
<td>Hamming</td>
<td>$\approx 2$</td>
<td>-41</td>
<td>-6</td>
</tr>
<tr>
<td>Blackman</td>
<td>$\approx 3$</td>
<td>-57</td>
<td>-18</td>
</tr>
</tbody>
</table>

- MLW – narrow for better spectral resolution
- PSL – lower to have less masking from nearby components
- SLR – “better” (faster) to have less masking from far away components

### Summary

Windowing of a simple waveform, like $\cos(\omega_0 t)$ causes its Fourier transform to have non-zero values (commonly called leakage) at frequencies other than $\omega_0$. It tends to be worst (highest) near $\omega_0$ and least at frequencies farthest from $\omega_0$. If there are two sinusoids, with different frequencies, leakage can interfere with the ability to distinguish them spectrally. If their frequencies are dissimilar, then the leakage interferes when one sinusoid is much smaller in amplitude than the other. That is, its spectral component can be hidden by the leakage from the larger component. But when the frequencies are near each other, the leakage can be sufficient to interfere even when the sinusoids are equal strength; that is, they become *unresolvable*. 
Conclusions

Selection of a window function for DFT must be done based on the application. A rectangular window is the choice when a high frequency resolution is desired. However, this type of window may be a bad pick if we expect signals in a wide dynamic range.

Questions?