Image smoothing - ILPF

An ideal low-pass filter (ILPF) is specified as

\[ H(u, v) = \begin{cases} 
1 & D(u, v) \leq D_0 \\
0 & D(u, v) > D_0 
\end{cases} \]

Here \( D_0 \) is a positive constant and \( D(u, v) \) is the distance between the origin and the point \((u, v)\) in the frequency domain:

\[ D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2} \]

\( P \) and \( Q \) are sizes of the zero-padded image.
Image smoothing - ILPF

$D_0$ is also called a **cut-off frequency**. One way to specify a cut-off frequency is by circles enclosing specific amount of total power $P_T$ in an image, where

$$P_T = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} P(u,v) = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} |F(u,v)|^2$$

If the DFT is centered, a circle of radius $D_0$ will enclose $\alpha$ percent of power:

$$\alpha = 100 \cdot \sum_u \sum_v P(u,v)/P_T$$

for the summation over the values being inside the circle or on its boundary.

Test image and its DFT

Test circles show ILPFs
The original image and results of filtering with circular ILPF of radii in pixels (enclosing % of total power): 10 (87), 30 (93.1), 60 (95.7), 160 (97.8), and 460 (99.2).

The filters removed 13, 6.9, 4.3, 2.2, and 0.8 \% of total power. Observe blurring and ringing.

A spatial representation of ILPF of radius 10. the center lobe causes blurring, the outer lobes introduces ringing...
Butterworth LPF

A transfer function of a Butterworth LPF (BLPF) of order $n$ and with cutoff frequency at the distance $D_0$ from the origin:

$$H(u,v) = \frac{1}{1 + \left[ \frac{D(u,v)}{D_0} \right]^{2n}}$$

Where:

$$D(u,v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}$$

$P$ and $Q$ are sizes of the zero-padded image.

Unlike the ILPF, the BLPF transfer function does not have sharp discontinuities.
No visible ringing and smooth transition between low and high frequencies.
Butterworth LPF

The original image and results of filtering with circular BLPF of order 2 and of radii in pixels (enclosing % of total power): 10 (87), 30 (93.1), 60 (95.7), 160 (97.8), and 460 (99.2).
The filters removed 13, 6.9, 4.3, 2.2, and 0.8 % of total power.
Observe no ringing.

BLPF of order 1 has neither ringing nor negative values; BLPF of order 1 has mid ringing and small negative values; while the order increases, ringing becomes more pronounced. For \( n = 20 \), BLPF is similar to ILPF. The second order is a good compromise between effective LP filtering and acceptable ringing.
Gaussian LPF (GLPF)

A transfer function of a Gaussian LPF (GLPF) with cutoff frequency at the distance $D_0$ from the origin:

$$H(u, v) = e^{-D^2(u, v)/2D_0^2}$$

Where:

$$D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}$$

$P$ and $Q$ are sizes of the zero-padded image.

The original image and results of filtering with circular GLPF of radii in pixels (enclosing % of total power): 10 (87), 30 (93.1), 60 (95.7), 160 (97.8), and 460 (99.2).

The filters removed 13, 6.9, 4.3, 2.2, and 0.8 % of total power.

Observe no ringing. Slightly less smoothing than BLPF.
Examples of smoothing

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

A text of low resolution: some characters are distorted, which may cause a problem for text recognizing software.

Result of filtering with a GLPF with $D_0 = 80$ (broken character segments were joined).

Original image showing "fine skin lines" around eyes.

Filtered with GLPF $D_0 = 100$

Filtered with GLPF $D_0 = 80$

Notice reduction in fine lines
Examples of smoothing

Original image: notice horizontal scan-lines

Smoothed with a GLPF with $D_0 = 50.$

Smoothed with a GLPF with $D_0 = 20.$

Image sharpening - HPF

A general HPF can be designed from a LPF as

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

Butterworth filter represents a transition between the sharpness of the ideal HPF and the broad smoothness of the Gaussian HPF.
Ideal HPF (IHPF)

An ideal high-pass filter (IHPF) is specified as

\[ H(u, v) = \begin{cases} 
0 & D(u, v) \leq D_0 \\
1 & D(u, v) > D_0 
\end{cases} \]

Here \( D_0 \) is a cut-off frequency and \( D(u, v) \) is the distance between the origin and the point \((u, v)\) in the frequency domain:

\[ D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2} \]

\( P \) and \( Q \) are sizes of the zero-padded image.
Ideal HPF (IHPF)

Filtered with IHPF with \( D_0 = 30 \)  
Filtered with IHPF with \( D_0 = 60 \)  
Filtered with IHPF with \( D_0 = 160 \)  

Images were made brighter for clarity. Observe severe ringing in first figure – distorted, thickened edges.

Butterworth HPF (BHPF)

A transfer function of a Butterworth HPF (BHPF) of order \( n \) and with cutoff frequency at the distance \( D_0 \) from the origin:

\[
H(u, v) = \frac{1}{1 + \left[D_0 / D(u, v)\right]^{2n}}
\]

Where:

\[
D(u, v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}
\]

\( P \) and \( Q \) are sizes of the zero-padded image.
Butterworth HPF (BHPF)

BHPF of order 2 were implemented. Observe no ringing and much less distortion than for IHPF.

Gaussian HPF (GHPF)

A transfer function of a Gaussian HPF (GHPF) with cutoff frequency at the distance $D_0$ from the origin:

$$H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

Where:

$$D(u,v) = \sqrt{(u-P/2)^2+(v-Q/2)^2}$$

$P$ and $Q$ are sizes of the zero-padded image.
Gaussian HPF (GHPF)

The results are more gradual than those obtained with IHPF and BHPF. Filtering of smaller objects is cleaner.

Examples of HP filtering

Original image of a thumb print with smudges (a typical problem in automated recognition); a result of Butterworth HPF with $D_0 = 50$: no gray tones; a result of thresholding: ridges are clearer, smudges are reduced.
Laplacian in frequency domain

Laplacian can be implemented as

\[ H(u, v) = -4\pi^2 (u^2 + v^2) = -4\pi^2 D^2(u, v) \]

Where \( D(u, v) \) is the distance function.

Therefore, the Laplacian image is

\[ \nabla^2 f(x, y) = \mathcal{F}^{-1}\{H(u, v)F(u, v)\} \]

and the enhanced image:

\[ g(x, y) = f(x, y) + c\nabla^2 f(x, y) \]

Since \( H(u, v) \) is negative, \( c = -1 \).

However, values of \( f \) and its Laplacian must be brought into the compatible ranges. The easiest way to do it would be: to scale \( f(x,y) \) to the range \([0,1]\) (then compute its DFT) and divide the Laplacian image \( \nabla^2 f(x,y) \) by its max value before applying.
Unsharp masking, High-boost and HF-Emphasis filtering

Unsharp masking/high-boost filtering can be implemented in the frequency domain also. The mask would be

\[ g_{\text{mask}}(x, y) = f(x, y) - f_{LP}(x, y) \]

Where a smoother image \( f_{LP}(x, y) = \mathcal{F}^{-1}\{H_{LP}(u, v)F(u, v)\} \)

Then, the filtered image is

\[ g(x, y) = f(x, y) + k \cdot g_{\text{mask}}(x, y) \]

This expression defines unsharp masking when \( k = 1 \) and high-boost filtering when \( k > 1 \)

Unsharp masking, High-boost and HF-Emphasis filtering

In the frequency domain, in terms of a LPF:

\[ g(x, y) = \mathcal{F}^{-1}\{[1 + k \cdot [1 - H_{LP}(u, v)]]F(u, v)\} \]

or in terms of a HPF:

\[ g(x, y) = \mathcal{F}^{-1}\{[1 + k \cdot H_{HP}(u, v)]F(u, v)\} \]

or more general:

\[ g(x, y) = \mathcal{F}^{-1}\{[k_1 + k_2 \cdot H_{HP}(u, v)]F(u, v)\} \]

A high-frequency-emphasis filter

Here, \( k_1 \geq 0 \) controls the offset from the origin (adds DC) and \( k_2 \geq 0 \) controls the contribution of high frequencies.
Unsharp masking, High-boost and HF-Emphasis filtering

An X-ray (slightly blurred and biased towards black) image;
Result of GHPF with $D_\theta = 40$: featureless gray;
Result of HF-emphasis filtering with $k_1 = 0.5$ and $k_2 = 0.75$: the image is dark but still has some details;
Result of histogram equalization of the HFE filtered image.

Homomorphic filtering

Since the illumination-reflectance model suggests that

$$f(x, y) = i(x, y)r(x, y)$$

and, defining:

$$z(x, y) = \ln \{ f(x, y) \} = \ln \{ i(x, y) \} + \ln \{ r(x, y) \}$$

Then:

$$\Im \{ z(x, y) \} = \Im \{ \ln [ f(x, y) ] \} = \Im \{ \ln [ i(x, y) ] \} + \Im \{ \ln [ r(x, y) ] \}$$

or

$$Z(u, v) = F_i(u, v) + F_r(u, v)$$

Filtering will lead to

$$S(u, v) = H(u, v)Z(u, v) = H(u, v)F_i(u, v) + H(u, v)F_r(u, v)$$

The filtered image will be

$$s(x, y) = \Im^{-1} \{ S(u, v) \} = \Im^{-1} \{ H(u, v)F_i(u, v) \} + \Im^{-1} \{ H(u, v)F_r(u, v) \} = i'(x, y) + r'(x, y)$$
Homomorphic filtering

Finally, since $z(x,y)$ was formed by a natural log, the filtered image:

$$g(x, y) = e^{s(x,y)} = e^{i(x,y)} e^{r(x,y)} = i_0(x, y)r_0(x, y)$$

This method is based on a special type of system – homomorphic systems. In this particular application, the key is to separate the illumination and reflectance components. The homomorphic filter $H(u,v)$ can operate on these components separately.

Homomorphic filtering

The illumination component of an image is generally characterized by slow special variations, while the reflectance component tends to vary abruptly. Therefore, we need a filter function $H(u,v)$ affecting low and high frequency components in different, controllable ways.

For instance:

$$H(u,v) = \left( \gamma_H - \gamma_L \right) \left[ 1 - e^{-c \cdot D^2(u,v)/D_0^2} \right] + \gamma_L$$

If $\gamma_L < 1$ and $\gamma_H > 1$, the filter will attenuate low frequencies (illumination) and amplify the high frequencies (reflectance). The net result is simultaneous dynamic range compression and constant enhancement.
Homomorphic filtering

Original PET image and its enhanced version with $\gamma_L = 0.25$, $\gamma_H = 2$, $c = 1$, $D_0 = 80$

Much sharper hot spots and more details.

Selective filters: band-reject and band-pass

Ideal band-reject filter:
$$H(u,v) = \begin{cases} 
0 & \text{if } D_0 - \frac{W}{2} \leq D(u,v) \leq D_0 + \frac{W}{2} \\
1 & \text{otherwise}
\end{cases}$$

Butterworth band-reject filter:
$$H(u,v) = \frac{1}{\left( 1 + \left( \frac{D(u,v) \cdot W}{D^2(u,v) - D_0^2} \right)^{2n} \right)}$$

Gaussian band-reject filter:
$$H(u,v) = 1 - e^{- \left( \frac{D^2(u,v) - D_0^2}{D^2(u,v) \cdot W} \right)^2}$$
Selective filters: band-reject and band-pass

A band-pass filter can be derived from a band-reject filter:

\[ H_{BP}(u, v) = 1 - H_{BR}(u, v) \]

Gaussian band-reject filter  Gaussian band-pass filter

Selective filters: band-reject and band-pass

Image corrupted with sinusoidal interference; its DFT with bursts of energy responsible for interference; filter used; filtered image.
Selective filters: notch

Notch filters are the most useful of the selective filters. Such filters reject (or pass) frequencies in a predefined neighborhood about the center of the frequency rectangle. Since zero-phase-shift filters must be symmetric about the origin, a notch with center at \((u_0, v_0)\) must have a corresponding notch at \((-u_0, -v_0)\).

A general notch reject filter is:

\[
H_{NR}(u, v) = \prod_{k=1}^{K} H_k(u, v)H_{-k}(u, v)
\]

Where \(H_k(u, v)\) and \(H_{-k}(u, v)\) are HFPs, whose centers are at \((u_k, v_k)\) and \((-u_k, -v_k)\) with respect to the center of frequency rectangle \((M/2, N/2)\).

The distance computations for each filter are:

\[
D_k(u, v) = \sqrt{(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2}
\]

For example, a Butterworth notch-reject filter of order \(n\) with 3 notch pairs:

\[
H_{NR}(u, v) = \prod_{i=1}^{3} \frac{1}{1 + \left[D_0/D_k(u, v)\right]^{2n}} \frac{1}{1 + \left[D_0/D_{-k}(u, v)\right]^{2n}}
\]

A notch-pass filter can be derived from a notch-reject filter:

\[
H_{NP}(u, v) = 1 - H_{NR}(u, v)
\]
Selective filters: notch

One of the principal applications of notch filtering is to selectively modify local regions of the DFT. This is typically done interactively working with DFT obtained without zero-padding. The advantages of working with actual DFTs (as opposed to having to “translate” from padded to actual frequency values) usually overweight any wraparound errors that may result from not using padding in the filtering process.
Selective filters: notch

Part of the Saturn’s rings showing nearly periodic interference;
DFT: the bursts of energy in the vertical axis near the origin correspond to the interference;
vertical notch-reject filter;
result of filtering.

Selective filters: notch

Application of a notch-pass filter (instead of a notch-stop) to the original image results in the spatial interference pattern itself to be preserved instead of the image...