Lecture 04: Analytic signal
generation and Hilbert transformers

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Analog analytic signal generation

All naturally generated signals are real-valued. However, some applications require
generation of a complex signal from a real input. One approach is by using a
Hilbert transformer characterized by an impulse response:

\[ h_{HT}(t) = \frac{1}{\pi t} \]  

(4.2.1)

whose CTFT is

\[ H_{HT}(j\Omega) = \begin{cases} -j, & \Omega > 0 \\ j, & \Omega < 0 \end{cases} \]  

(4.2.2)

Considering a real-valued analog signal \( x(t) \). The magnitude spectrum of \( x(t) \) –
since the signal is real – must have an even symmetry, while the phase spectrum
has an odd symmetry. Therefore, the spectrum is:

\[ X(j\Omega) = X_p(j\Omega) + X_n(j\Omega) \]  

(4.2.3)
Analog analytic signal generation

If \( x(t) \) passes through a Hilbert transformer, the spectrum of the output signal is:
\[
\tilde{X}(j\Omega) = H_{HT}(j\Omega)X(j\Omega) = -jX_p(j\Omega) + j\hat{X}(j\Omega) \tag{4.3.1}
\]
It can be shown that \( \hat{x}(t) \) is also a real signal.
Consider the signal \( y(t) \) formed as
\[
y(t) = x(t) + j\hat{x}(t)
\]
Its CTFT can be expressed as:
\[
Y(j\Omega) = X(j\Omega) + j\hat{X}(j\Omega) = 2X_p(j\Omega) \tag{4.3.3}
\]
Therefore, the complex signal called an analytic signal has only the positive frequency components.

Digital analytic signal generation

A discrete-time analytic signal has the same property as the analog one. It is frequently used in single-sideband communication systems, for example.

The DTFT \( X(e^{j\omega}) \) of a real \( x_n \) — if exist — is nonzero for both positive and negative frequencies. If \( \hat{x}_n \) is a complex analytic signal generated from \( x_n \) it must have a single-sided spectrum \( Y(e^{j\omega}) \). In other words:
\[
y_n = x_n + j\hat{x}_n \tag{4.4.1}
\]
\[
Y(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega}) \tag{4.4.2}
\]
Since both \( x_n \) and \( \hat{x}_n \) are real, their DTFTs are conjugate symmetric:
\[
X(e^{j\omega}) = X^*(e^{-j\omega}) \quad \text{and} \quad \hat{X}(e^{j\omega}) = \hat{X}^*(e^{-j\omega}) \tag{4.4.3}
\]
Therefore:
\[
X(e^{j\omega}) = \frac{1}{2} \left[ Y(e^{j\omega}) + Y^*(e^{-j\omega}) \right] \tag{4.4.4}
\]
\[
j\hat{X}(e^{j\omega}) = \frac{1}{2} \left[ Y(e^{j\omega}) - Y^*(e^{-j\omega}) \right] \tag{4.4.5}
\]
Digital analytic signal generation

Since, for an analytic signal
\[ Y(e^{j\omega}) = \begin{cases} 2Y(e^{j\omega}), & 0 \leq \omega < \pi \\ 0, & -\pi \leq \omega < 0 \end{cases} \] (4.5.1)
such a signal can be generated by passing \( x_n \) through a linear filter with a frequency response
\[ H(e^{j\omega}) = \begin{cases} 2, & 0 \leq \omega < \pi \\ 0, & -\pi \leq \omega < 0 \end{cases} \] (4.5.2)
that is shown

Digital AS generation: Hilbert Transformer

The imaginary part of an analytic signal should have a spectrum
\[ \hat{X}(e^{j\omega}) = \frac{1}{2j} \left[ Y(e^{j\omega}) - Y^*(e^{-j\omega}) \right] \] (4.6.1)
Therefore, using (4.5.1), we derive
\[ \hat{X}(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}), & 0 \leq \omega < \pi \\ jX(e^{j\omega}), & -\pi \leq \omega < 0 \end{cases} \] (4.6.2)
Which indicates that the imaginary part of the analytic signal can be generated by passing its real part \( x_n \) through a linear system with the frequency response:
\[ H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \leq \omega < \pi \\ j, & -\pi \leq \omega < 0 \end{cases} = -j \text{sign}(\omega) \] (4.6.3)
Digital AS generation: Hilbert Transformer

A linear system with that frequency response is called an ideal Hilbert transformer and its output is called the Hilbert transform of \( x[n] \).

We observe that \( |H_{HT}(e^{j\omega})| = 1 \) for all frequencies and has a -90° phase shift for positive frequencies and a +90° phase shift for negative frequencies. Therefore, an ideal Hilbert transformer is also called a 90-degree phase shifter.

The impulse response of the ideal Hilbert transformer is

\[
H_{HT,n} = \begin{cases} 
0, & \text{for even } n \\
\frac{2}{\pi n}, & \text{for odd } n
\end{cases}
\] (4.7.1)

Since the ideal Hilbert transformer has a two-sided (i.e., non-causal) infinite-length impulse response, such a system is unrealizable.

Digital AS generation: relation with Half-band filters

We consider a filter with the frequency response \( G(e^{j\omega}) \) obtained by shifting the frequency response in (4.5.2) by \( \pi/2 \) radians and scaling by a factor \( 1/2 \):

\[
G(e^{j\omega}) = \frac{1}{2} G(e^{j(\omega + \pi/2)}) = \begin{cases} 
1, & 0 < |\omega| < \pi/2 \\
0, & \pi/2 < |\omega| < \pi
\end{cases}
\] (4.8.1)

\( G(e^{j\omega}) \) is a half-band LPF. Therefore, the filter \( H(e^{j\omega}) \) – the filter that is needed to generate an analytic signal – is called a complex half-band filter.
Digital AS generation: design of Hilbert transformer

It follows from the previous discussion that a complex half-band filter \( H(z) \) can be designed by modifying the transfer function of a real half-band filter \( G(z) \).

\[
H(z) = j2G(-jz) \tag{4.9.1}
\]

We consider two methods to design complex half-band filters.

1. FIR complex half-band filter

Let \( G(z) \) be the real FIR half-band linear phase LPF of even order \( N-1 \) with the passband and stopband frequencies of \( \omega_p \) and \( \omega_s \), where \( \omega_p + \omega_s = \pi \), and passband and stopband ripples of \( \delta \).

To design such a half-band filter, we first generate a wide-band linear phase filter \( F(z) \) of order \( (N-1)/2 \) with a passband from 0 to \( 2\omega_p \) a transition band from \( 2\omega_p \) to \( \pi \) and a passband ripple of \( 2\delta \). Then:

\[
G(z) = \frac{1}{2} \left[ z^{-[(N-1)/2]} + F\left(z^2\right) \right] \tag{4.9.2}
\]

Note: an FIR Hilbert transformer can be designed directly by a ML functions \texttt{firpm} or \texttt{firls}.

Digital AS generation: design of Hilbert transformer

Then, a complex half-band filter can be obtained as

\[
H(z) = j \left[ (-jz)^{-[(N-1)/2]} + F\left(z^2\right) \right] = z^{-[(N-1)/2]} + jF\left(-z^2\right) \tag{4.10.1}
\]

We notice that \( F(-z^2) \) is an FIR approximation to a Hilbert transformer.
2. IIR complex half-band filter

A BIBO stable IIR real coefficient half-band filter can be designed as

\[ G(z) = \frac{1}{2} \left[ A_0(z^2) + z^{-1} A_1(z^2) \right] \]  

(4.11.1)

Where \( A_0(z) \) and \( A_1(z) \) are stable allpass transfer functions. Therefore, a complex half-band filter will be

\[ H(z) = A_0(-z^2) + jz^{-1} A_1(-z^2) \]  

(4.11.2)

Normalized gain (magnitude) response of the complex half-band IIR filter

Phase difference between the two allpass sections of the complex half-band IIR filter
Digital AS implementation: single-sideband modulation

To transmit a real low-frequency band-limited signal \( x_n \) over long distances, it is modulated by a very high frequency sinusoidal carrier \( \cos \omega_c n \) of the frequency \( \omega_c \) that is less than half of the sampling frequency. The resulting signal

\[
V_n = x_n \cos \omega_c n
\]  

will have the spectrum

\[
V(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j(\omega - \omega_c)}) + X(e^{j(\omega + \omega_c)}) \right]
\]  

which is symmetric with respect to the carrier frequency.

Therefore, the portion of the spectrum in the frequency range from \( \omega_c \) to \( \omega_c + \omega_M \) (upper sideband) has the same information content as the portion of the frequency range from \( \omega_c - \omega_M \) to \( \omega_c \) (lower sideband). For a more efficient utilization of the channel bandwidth, it is sufficient to transmit either the upper or lower sideband signal.

One way to eliminate one of the sidebands is to pass the modulated signal \( v_n \) through a sideband filter, whose passband covers the frequency range of one of the sidebands.

An alternative, often preferred, approach for single-sideband signal generation is by modulating the analytic signal obtained by a Hilbert transformer. Let

\[
Y_n = x_n + j\hat{x}_n
\]  

be the analytic signal generated from a real signal \( x_n \).
Digital AS implementation: single-sideband modulation

Consider

\[ s_n = y_n e^{j\omega_c n} = (y_{re,n} + jy_{im,n})(\cos \omega_c n + j \sin \omega_c n) \]
\[ = \left[ x_n \cos \omega_c n - \hat{x}_n \sin \omega_c n \right] + j \left[ x_n \sin \omega_c n + \hat{x}_n \cos \omega_c n \right] \tag{4.15.1} \]

The real and imaginary parts of the modulated signal are specified by

\[ s_{re,n} = x_n \cos \omega_c n - \hat{x}_n \sin \omega_c n \tag{4.15.2} \]
\[ s_{im,n} = x_n \sin \omega_c n + \hat{x}_n \cos \omega_c n \tag{4.15.3} \]

A single-sideband signal can be generated using either real (4.15.2) or imaginary (4.15.3) part.
Digital AS implementation: single-sideband modulation

Spectrum of the entire modulated analytic signal

Spectrum of the real part of the modulated analytic signal

Spectrum of the imaginary part of the modulated analytic signal

A diagram of a single-side generator utilizing a real part of the modulated analytic signal.

A diagram of a single-side generator utilizing a real part of the modulated analytic signal.