Narration and chalkboard discussion to the Mathematics Seminar
by Bernard Maxum
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on the work of and in collaboration with Dr. Bengt Fornberg, et. al. at the University of Colorado, Boulder

Slide	NARRATION
1	Title page: "Recent developments with the use of RBFs for solving PDEs" This is an brief synopsis of the progress, current status and future developments of work being conducted by Dr. Bengt Fornberg and his team of collaborators and students at the University of Colorado—Boulder. These comments and observations are primarily based on my association with Dr. Fornberg and his team during four brief visits over the past two semesters.
2	Introduction. Over the past twenty years pseudospectral (PS) methods have emerged as powerful numerical tools for obtaining spectrally accurate solutions to partial differential equations on simple geometries. PS methods proved to be superior to conventional methods, such as finite difference (FD) and finite element (FE) methods in efficiency and accuracy. PS methods, however, have inherent within them two shortcomings. The first is that, like FD and FE, there is no assurance that the PS system is nonsingular for all node choices. Furthermore, PS methods are confined to regular coordinate-based geometries. I. Schoenberg showed that for positive definite radial functions $\phi(r)$, matrices with entries $A_{ij} = \left[\phi\left(\left\|\underline{x}_i - \underline{x}_j\right\|\right)\right]$ are non-singular for all node distributions, thereby addressing the first limitation. The second is addressed by Fornberg and his team of collaborators and students at the University of Colorado who have shown that radial basis function (RBF) methods offer outstanding geometric flexibility. In this seminar I will outline comparative numerical experiments involving RBF methods conducted at the University of Colorado over the past two semesters, present some results from these experiments, and discuss some of their future projects.

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3	Contents: Four topics are covered. First we summarize the power of the PS method over the historical finite difference FD and finite element FE methods. We find that some of the limitations of the FD and FE methods also remain with the PS method. Then we introduce radial basis functions (RBFs) that can alleviate these limiting issues associated with PS methods. The procedure involves setting up an appropriate RBF interpolant and finding the combination of rotated versions of the RBF that collate the data. This is accomplished by first creating a "Poisson sum", that is the summation of the displaced RBFs for each node in Fourier space, then constructing the ratio of each of the components to the Poisson sum . Then taking the inverse transform of that ratio we obtain the RBF spectral filter function. Finally, by applying the various weights from the original data the cardinal interpolant that collates the data is determined. The above described procedure is first formulated for the one-dimensional (1-D) case where the interpolant *c(x)* collates the data $f_i\left(x_i\right)$ and single Fourier transforms are taken. This is followed by two 2-D cases where *c(x, y)* collates the data $f_i\left(x_i, y_i\right)$ and where double FTs are taken. Two 2-D cases are studied—one for Cartesian Lattices, the other for hexagonal lattices. Of the several numerical experiments conducted the comparisons between the Cartesian and hexagonal lattices are summarize in this seminar.
4	PS Method—its power and limitations. Being a higher order FD method, SP methods retain all of the benefits of the historically proven FD method but with greater efficiency and accuracy. However the PS approach is generally restricted to simple geometries such as spheres where derivatives are restricted to coordinate directions and therefore fall short for applications that lend themselves to generalized surfaces, such as facial recognition or brain malady studies. Such problems are illustrated by the simple illustration that follows.
5	PS Stencils for $\partial/\partial r$. Suppose we have a function $f(x, y)$ and ask for a derivative along the line $x = y$. By conventional FD and even the more powerful PS methods the quantity $\frac{\partial f}{\partial r} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f(x, y)$ evaluated at the center point is typically used to approximate the derivative along the line $x = y$. Using the data values at the nodes marked "x", the partial derivative $\frac{\partial f}{\partial x}$ is evaluated. Likewise $\frac{\partial f}{\partial y}$ is approximated by the ones marked "o". However, the desired derivative is in the direction marked by the "boxes". But none of these nodes were used. A derivative is a local property of a function and yet none of the nodes along the true direction are used.

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6	This figure depicts <u>nine- and five-node stencils for approximating $\partial f/\partial r$</u> along the line $x = y$. The circles denote the typical nine nodes used in the approximation; whereas, five nodes denoted as squares along the $x = y$ line are all that would be needed to for the desired derivative if the numerical methodology used would allow off axis nodes. This simple example illustrates a fundamental shortcoming of computational efficiency (minimal computer running time) for a given accuracy when using FD and even PS methods. The solution to these shortcomings lies with the use of a special set of basis functions, especially radial basis function (RBFs).
7	<u>Introduction to Radial Basis Functions</u>. RBF approximations are global in nature and combine the information from all node points, thus acquiring information that FD and PS ignore. With a radial function $\phi(r)$ and with data values f_k given at locations x_k, the function $s(x)=\sum_{k=1}^{n}\lambda_k\phi(\Vert \underline{x}-\underline{x}_k\Vert)$ interpolates the data if we choose the expansion coefficients λ_k such that $s(x_k)=f_k$ for each node $k=1,2,\cdots,n$ where $\Vert\ \Vert$ denotes the Euclidian norm. The expansion coefficients are obtained by solving the linear system $\underline{\underline{A}}\cdot\underline{\lambda}=\underline{f}$ where $A_{ij}=\phi(\Vert \underline{x}_i-\underline{x}_j\Vert)$. The class of radial functions $\phi(r)$ that are infinitely differentiable and have positive definite Fourier transforms, when used in the form $A_{ij}=\phi(\Vert \underline{x}_i-\underline{x}_j\Vert)$ are nonsingular no mater how the nodes $\underline{x}_k$ are scattered. These are radial basis functions (RBFs). [Schoenberg 1938]
8	<u>Some RBFs and their Transforms</u>. This chart tabulates four of the common proven RBFs and their transforms. The inverse functions IMQ and IQ and the Gaussian GA RBFs fall off for increasing *r*; however, even the cup-type MQ function works well as a suitable RBF cardinal interpolant. The Fourier transforms of these are shown in the two right-hand columns. The one dimensional forms require the single transform over the single Euclidian dimension *x*. The two-dimensional forms require double transforms over the dual Euclidian dimensions *x* and *y*. The inverse quadratic IQ RBF was used to illustrate the global utility of RBFs over PS and its predecessor methods in this seminar; however, as stated before, the efficiency and accuracy of the numerical results are similar when any of the RBFs that meet Schoenberg's criteria are used. Notice that each of the RBFs contain a shaping parameter ε. As $\varepsilon\to 0$ the RBF $\phi(r)$ flattens thereby providing several simplifications as well as numerical challenges. The FTs simplify as shown and the RBF spectral filter functions sharpen forming visual picture of the condition $\hat{\phi}(\rho)=1$ within the bounds of the periodic structure and zero outside of those bounds.

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9	What is meant by RBF interpolation? In the next series of charts (Charts 9 through 16) we reiterate what is meant by RBF interpolation and how the RBF interpolant is determined. In a sense this is a clarification of what has already been summarized. The shape of several radial basis functions is shown. Next the 1-D or 2-D data sets is depicted in the lower graphs illustrating a random distribution of values of f_k given at locations x_k in the 1-D example, or f_k at locations (x_k, y_k) in the 2-D case.
10	1-D Cardinal interpolant for the first node. This chart shows the 1-D cardinal node interpolant for the *k=1* data point. This particular interpolant is for the IQ RBF shown in red. The peak value of $\phi(x_1)$ is held at unity.
11	1-D Cardinal interpolant for five data points. This chart shows the 1-D cardinal interpolant for the first five nodes *k=1, 2, 3, 4,* and *5*, again for the IQ RBFs shown in red. This process continues until all n nodes are assigned their respective IQ RBF interpolants for $k = 1, 2, \cdots, n$. All peaks of $\phi(x_k)$ are unity.
12	Calculation of the RBF Interpolant. This chart outlines the setup of the calculation of the RBF interpolant $s(x)$ in 1-D, which is a continuous function $s(x) = \sum_{k=1}^{n} \lambda_k \phi(\\|x - x_k\\|)$, where the weighting factors or eigenvalues λ_k are determined by forcing *n* values of $s(x_k)$ to be equal to the known values of f_k and by solving the linear system $\underline{\underline{A}} \cdot \underline{\lambda} = \underline{f}$ $$\begin{bmatrix} \phi(\\|x_0 - x_0\\|) & \phi(\\|x_0 - x_1\\|) & \cdots & \phi(\\|x_0 - x_n\\|) \\\\ \phi(\\|x_1 - x_0\\|) & \phi(\\|x_1 - x_1\\|) & \cdots & \phi(\\|x_1 - x_n\\|) \\\\ \vdots & \vdots & \ddots & \vdots \\\\ \phi(\\|x_n - x_0\\|) & \phi(\\|x_n - x_1\\|) & \cdots & \phi(\\|x_n - x_n\\|) \end{bmatrix} \bullet \begin{bmatrix} \lambda_0 \\\\ \lambda_1 \\\\ \vdots \\\\ \lambda_n \end{bmatrix} = \begin{bmatrix} f_0 \\\\ f_1 \\\\ \vdots \\\\ f_n \end{bmatrix}$$ for the λ_ks, where $A_{ij} = \phi(\\|\underline{x}_i - \underline{x}_j\\|)$. Each component RBF $\phi(\\|x_i - x_j\\|)$ is known for each value of x_i and x_j. Also, the $s(x_k) = f_k$ values are known. Because of Schoenberg's theorem (1938), assuring the nonsingularity of $[\underline{\underline{A}}]$ for radial functions that are infinitely differentiable and have positive definite Fourier transforms, the eigenvalues, λ_k, are determinable.
13	1-D RBF Interpolant that collates the data. The detailed process for finding the resulting smooth interpolating function *s(x)* that fits the original data set $f_k = s(x_k)$ at each node $k = 1, 2, \cdots, n$ as shown by the red curve in this slide, is described in slides 17 through 21. But first we repeat this 1-D setup for the 2-D case. The 2-D setup is outlined in the next three slides.

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14	2-D Cardinal Interpolants for five data points. Two-dimensional cardinal interpolants are illustrated for five randomly selected nodes. These are rotations of the IQ RBF shown in slide 10, where the axis of rotation is perpendicular to the *x, y* plane. Again, each interpolant is of unity height.
15	Calculation of the 2-D RBF Interpolant. This chart outlines the setup of the calculation of the RBF interpolant $s(\underline{x}) = s(x,y)$ in 2-D, which is a continuous function $\sum_{k=1}^{n} \lambda_k \phi(\lVert \underline{x} - \underline{x}_k \rVert) = \sum_{k=1}^{n} \lambda_k \phi\left(\left\lVert \sqrt{(x-x_k)^2 + (y-y_k)^2} \right\rVert\right)$, where the weighting factors or eigenvalues λ_k are determined by forcing *n* values of $s(x_k,\ y_k) = f_k,\quad k = 1, 2, \cdots, n$ to be equal to the known values of f_k by solving the linear system $\underline{\underline{A}} \cdot \underline{\lambda} = \underline{f}$ $$\begin{bmatrix} \phi(\lVert \underline{x}_0 - \underline{x}_0 \rVert) & \phi(\lVert \underline{x}_0 - \underline{x}_1 \rVert) & \cdots & \phi(\lVert \underline{x}_0 - \underline{x}_n \rVert) \\ \phi(\lVert \underline{x}_1 - \underline{x}_0 \rVert) & \phi(\lVert \underline{x}_1 - \underline{x}_1 \rVert) & \cdots & \phi(\lVert \underline{x}_1 - \underline{x}_n \rVert) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\lVert \underline{x}_n - \underline{x}_0 \rVert) & \phi(\lVert \underline{x}_n - \underline{x}_1 \rVert) & \cdots & \phi(\lVert \underline{x}_n - \underline{x}_n \rVert) \end{bmatrix} \bullet \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$ for the λ_ks, where $A_{ij} = \phi\left(\left\lVert \underline{x}_i - \underline{x}_j \right\rVert\right)$. Each component RBF $\phi\left(\left\lVert \underline{x}_i - \underline{x}_j \right\rVert\right)$ is known for each known value of $\underline{x}_i = (x_i, y_i)$ and $\underline{x}_j = (x_j, y_j)$. Because Schoenberg's theorem applies for any dimensionality, the eigenvalues λ_k are determinable.
16	2-D RBF Interpolant that collates the data. The detailed process for finding the resulting smooth interpolating function *s(x, y)* that fits the original data set $f_k = s(x_k, y_k)$ at each node $k = 1, 2, \cdots, n$ as shown by the continuous surface in this slide, is described in Slides 23. through 37 for uniform lattice structures.
17	RBF Studies. Now that we have set up the methodology for obtaining the 1-D and 2-D RBF interpolants, we next sketch some of the numerical experiments that were conducted. Although the methodology is applicable to random scattering of nodes in 1-D or 2-D, we will discuss only those 1-D and 2-D studies that involved infinite uniform lattices, that is, with the nodes equally spaced and extending from $-\infty$ to $+\infty$. Slides 18 through 21 summarize the 1-D case. Two 2-D cases are covered, namely, where the lattices are in Cartesian and hexagonal configurations. With the Cartesian-lattice case nodes are located at $(x_i, y_j);\ i, j = -\infty, \cdots, -1, 0, 1, \cdots, \infty$. The infinite hexagonal lattice involves closer stacking like bowling pins. A comparison is made between these two lattice structures demonstrating the power and accuracy of RBF methods and illustrating the advantage of the closer stacking of the hexagonal lattice.

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18	Poisson sum and RBF spectral filter functions for the 1-D case. For these numerical experiments, we chose to use the Inverse Quadratic (IQ) RBF $\phi(r,\varepsilon)=1\big/\left[1+(\varepsilon r)^2\right]$ in Euclidian (physical) space, where $r=x$. The single (1-D) Fourier transform of the IQ RBF $\hat{\phi}(\xi,\varepsilon)$ is then determined as $\hat{\phi}(\xi,\varepsilon)=\lvert\rho\rvert^{1/2}K_{-1/2}\left(\frac{\lvert\rho\rvert}{\varepsilon}\right)\big/\varepsilon^{3/2}$ where $\rho=\xi$, and where $K_\nu(z)$ is the modified Bessell function of the second kind of order ν and argument *z*. A "Poisson Sum" in Fourier space defined by $\Xi(\boldsymbol{\xi},\varepsilon)=\sum_{k=-\infty}^{\infty}\hat{\phi}(\boldsymbol{\xi}+2\pi k,\varepsilon)$ is next determined by summing each of the Fourier transforms of the uniformly spaced RBFs at each node. The 1-D Spectral filter function is a normalization of the FT'd RBF defined by the ratio $\hat{\phi}(\xi,\varepsilon)\big/\Xi(\xi,\varepsilon)$. As such it tends to 1 in the region $-\pi<\boldsymbol{\xi}<\pi$, and zero otherwise as the shape parameter ε tends to zero.
19	1-D RBF spectral filter functions for uniform infinite nodes. The one-dimensional spectral filter function is plotted for several values of the shape parameter ranging from $\varepsilon=1.0$ to $\varepsilon=0.05$. One can readily see that as $\varepsilon\to 0$, the spectral filter function (in Fourier Space) takes the value of one in the range $-\pi<\xi<\pi$ and zero otherwise.
20	The inverse Fourier transform and the 1-D cardinal interpolant for the *x=0* node. The 1-D inverse Fourier transform of the RBF spectral filter function for the k^{th} node is $s_k(x)=(1/2\pi)\int_{-\infty}^{\infty}\left(\hat{\phi}_k(\xi,\varepsilon)\big/\Xi(\xi,\varepsilon)\right)e^{i\xi\lvert x-x_k\rvert};\ k=0,\pm1,\pm2,\cdots$ which approaches $\to\frac{1}{2\pi}\int_{-\infty}^{\infty}(1)e^{i\xi\lvert x-x_k\rvert}d\xi;$ as $\varepsilon\to 0$. This reduces to $\mathrm{sinc}(\lvert x-x_k\rvert)$. For the $k=0^{th}$ node, $x_0=0$, and $s_0(x)=\mathrm{sinc}(\lvert x\rvert)$ is the 1-D cardinal interpolant shown on the right. All of the other cardinal interpolants are the same except they are displaced in *x* by $x_k=k$.
21	1-D RBF interpolant that collates the data is now determinable from the weighted summation of the RBF interpolants $s(x)=\sum_{k=-\infty}^{\infty}\lambda_k s_k(x)$ The process of obtaining the weighting factors λ_k was described in Chart 12.

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22	2-D RBF studies. Now that we have explained the process for determining the 1-D RBF interpolant for the case of uniform infinite nodes, we next cover the 2-D case. We will first describe the 2-D infinite lattice with Cartesian stacking.
23	The 2-D IQ RBF and its Fourier transform. For the 2-D cases that follow we use the same inverse quadratic IQ RBF as in the 1-D case, except r is the radial distance $r=\sqrt{x^2+y^2}$. However, we must take double spatial Fourier transforms over *x,y* Euclidian (physical) space, which transforms us into ξ,η Fourier space. This double transform actually yields a more simple result $\hat{\phi}(\xi,\eta,\varepsilon)=K_0\left(\lvert\rho\rvert/\varepsilon\right)/\varepsilon^2$, where $\rho=\sqrt{\xi^2+\eta^2}$ and K_0 is the modified Bessell function of the second kind of order 0.
24	Poisson sum and RBF spectral filter functions for the 2-D infinite uniform lattice with **Cartesian** stacking. In this case the Poisson sum consists of the double summation of the Fourier transformed RBFs each displaced by 2π in each direction in Fourier space. Since this obvious 2π repetition does not take place in the next (hexagonal stacking) case, we designate the Poisson sum with subscript "Cart" to emphasize the Cartesian stacking. The spectral filter function is also so designated because it is the normalized FT'd RBF for each node. The arguments of the functions explicitly call out the two-dimensional Fourier space variables ξ,η and the shape parameter ε.
25	2-D Poisson sum and RBF spectral filter function for Cartesian lattice. Graphical representations of the 2-D uniform Cartesian lattice Poisson sum (left) and the corresponding RBF spectral filter function (right) are shown. The characteristic Cartesian stacking with uniform spacing in both ξ and η is clearly illustrated, corresponding to the uniform stacking in physical *x,y* space. The spectral filter function, corresponding to the ratio $R_{Cart}(\xi,\eta,\varepsilon)=\hat{\phi}_{Cart}(\xi,\eta,\varepsilon)/\Xi_{Cart}(\xi,\eta,\varepsilon)$ tends to 1 in the rectangular region $R_{Cart}(\xi,\eta)=1;\ -\pi<\xi,\eta<\pi$ and 0 elsewhere as $\varepsilon\to 0$.
26	The 2-D cardinal interpolant for the $\overline{k\ell}^{th}$ node in an infinite Cartesian lattice. By taking the double spatial inverse Fourier transform of the spectral filter function for the $\overline{k\ell}^{th}$ node we obtain that node's cardinal interpolant $s_{k\ell}(x,y)$ for the case of the infinite uniform Cartesian lattice. However, since the spectral filter function $R(\xi,\eta)=1$ in the square region $-\pi<\xi,\eta<\pi$ and zero elsewhere as $\varepsilon\to 0$, the ratio in the integrand is one, the integration limits range from $-\pi$ to π in both ξ and η Fourier space coordinates and the double integral becomes the product of two sinc functions $\mathrm{s}_{k\ell}(\mathrm{x},\mathrm{y})=\mathrm{sinc}\left(\pi\lvert\mathrm{x}-\mathrm{x}_k\rvert\right)\mathrm{sinc}\left(\pi\lvert\mathrm{y}-\mathrm{y}_\ell\rvert\right)$.

$\varepsilon = 2.0, 1.0, 0.5$

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27	2-D Cardinal interpolant for the 00-node on a Cartesian lattice. The cardinal interpolant $s_{k\ell}(x, y)$ for $k = \ell = 0$, that is for the $\overline{00}^{th}$ node, becomes $s_{00}(x, y) = \mathrm{sinc}(\pi x)\mathrm{sinc}(\pi y)$. Using the Mathematica "Plot3D" function, we obtain this plot of the $\overline{00}^{th}$ cardinal interpolant for the doubly infinite uniform Cartesian lattice. It rises to a value of one at $x = 0, y = 0$, as it must.
28	Zero contour lines for 2-D Cartesian lattice. Using the Mathematica function "ContourPlot" the zero contour lines of the cardinal interpolant $s_{00}(x, y)$, shown in the previous slide are plotted here. This is to be compared to the zero-contour plots for the 2-D hexagonal lattice shown in slide 36.
29	2-D RBF interpolant for 169-node Cartesian Stacking that collates the data is then determined by the double summation of the cardinal interpolants $s_{k\ell}(x, y)$ weighted by the double set of eigenvalues $\lambda_{k\ell}$ determined from the process described in Chart 15. The 13 x 13 Cartesian-stacked nodes are shown on the right.
30	The next 2-D RBF study is for the case of the nodes being stacked more tightly in a hexagonal pattern. We will find that this closer stacking leads to improved numerical accuracy.
31	The 2-D IQ RBF and its Fourier transform. The RBF and its Fourier transform are independent of the node configuration, so this slide for the 2-D inverse quadratic RBF, applicable to the hexagonal node pattern, is the same as Slide 23 that we showed before for the Cartesian pattern.
32	Poisson sum and RBF spectral filter functions for the 2-D infinite uniform lattice with **hexagonal** stacking. The Poisson sum has the same double summation form over k and ℓ indices as before, except the arguments of the FT'd RBF are taken at quite different points in Fourier space, namely $\xi + 4\pi(k + \ell/2)$ and $\eta + 2\pi\ell/\sqrt{3}$. The spectral filter function is defined as the ratio of the $\overline{k\ell}^{th}$ RBF in Fourier space to the Poisson sum as before, except for the reconfigured Fourier points.
33	2-D RBF spectral filter function for **hexagonal** lattice. This slide graphically presents the 2-D RBF spectral Filter function for the hexagonal lattice for three values of the shaping parameter, $\varepsilon = 2.0, 1.0, 0.5$ from left to right. Each peaks at unity and their hexagonal nature becomes more apparent at the smaller ε values.
34	An enlarged view of the spectral filter function for the $\varepsilon = 0.5$ case is shown in this chart. Notice the distinct hexagonal shape to the function and the approximation that the ratio equals unity in the hexagonal region and zero elsewhere. Notice that the points of the hexagon lie on the ξ axis.

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35	The 2-D cardinal interpolant for the $\overline{k\ell}^{th}$ node in an infinite hexagonal lattice. As with the Cartesian lattice case, the double integral form for the inverse Fourier transform over the spectral filter function for the $\overline{k\ell}^{th}$ node is essentially the same, except for the factor $\sqrt{3}/2$ to compensate for the more dense packing of the hexagonal configuration in order to maintain the same node density per unit area for numerical comparisons, in order to obtain that node's cardinal interpolant $s_{k\ell}(x, y)$. However, since the spectral filter function $R(\xi, \eta) = 1$ in the hexagonal region in Fourier space and zero elsewhere as $\varepsilon \to 0$, the ratio in the integrand can be replaced by one and the integration taken over the hexagon cross section. The double integral then takes the form of the second line indicated by $\frac{\sqrt{3}}{8\pi^2} \iint_{Hex} (1) e^{i(\xi\|x-x_k\|+\eta\|y-y_\ell\|)} d\xi\, d\eta$ as $\varepsilon \to 0$, where the integration region denoted by "Hex" is defined by the six points in Fourier space: $(4\pi/3, 0)$, $(2\pi/3, 2\pi/\sqrt{3})$, $(-2\pi/3, 2\pi/\sqrt{3})$, $(-4\pi/3, 0)$, $(-2\pi/3, -2\pi/\sqrt{3})$, and $(2\pi/3, -2\pi/\sqrt{3})$. Notice that the first ad fourth points lie on the ξ axis. It is convenient to break the $\iint_{Hex}$ into two half hexagons $\iint_{HexUpper}$ + $\iint_{HexLower}$. Therefore as the integration in η ranges from zero to $2\pi/\sqrt{3}$ for the upper half hexagon and from $-2\pi/\sqrt{3}$ to zero for the lower half hexagon, the ξ integration limits vary as a function of η. The upper half-hexagon limits vary from $(-4\pi/3 + \eta/\sqrt{3})$ to $(4\pi/3 - \eta/\sqrt{3})$ and the lower half-hexagon limits vary from $(-4\pi/3 - \eta/\sqrt{3})$ to $(4\pi/3 + \eta/\sqrt{3})$ as shown on the third line of the chart. Finally, after performing these integrations and factoring, we obtain $$s_{Hex_{k\ell}}(x,y) = \frac{3}{4\pi^2} \frac{-2x \cos\frac{4\pi\|x-x_k\|}{3} + (\|x-x_k\| - \sqrt{3}\|y-y_\ell\|)\cos\left[\frac{2\pi}{3}(\|x-x_k\| - \sqrt{3}\|y-y_\ell\|)\right] + (\|x-x_k\| + \sqrt{3}\|y-y_\ell\|)\cos\left[\frac{2\pi}{3}(\|x-x_k\| + \sqrt{3}\|y-y_\ell\|)\right]}{\|x-x_k\|(\|x-x_k\|^2 - 3\|x-x_k\|\|y-y_\ell\|^2)}$$ for the limiting 2-D cardinal interpolant for the $\overline{k\ell}^{th}$ node on an infinite hexagonal lattice as $\varepsilon \to 0$.
36	The limiting 2-D cardinal interpolant for the $\overline{00}^{th}$ (centered) node on an infinite hexagonal lattice as $\varepsilon \to 0$. The centered $\overline{00}^{th}$ node is found by setting $k = \ell = 0$ and $x_k = y_\ell = 0$ in the previous chart, which yields $$s_{Hex_{00}}(x, y) = \frac{3}{4\pi^2} \frac{-2x \cos\frac{4\pi x}{3} + (x - \sqrt{3}y)\cos\left[\frac{2\pi}{3}(x - \sqrt{3}y)\right] + (x + \sqrt{3}y)\cos\left[\frac{2\pi}{3}(x + \sqrt{3}y)\right]}{x(x^2 - 3xy^2)}$$ for the $\overline{00}^{th}$ limiting 2-D cardinal interpolant on an infinite hexagonal lattice as $\varepsilon \to 0$.

$\varepsilon \to 0$

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37	2-D hexagonal-lattice RBF spectral-filter functions, interpolants and zero-contour plots. The RBF spectral-filter-function, also called the Fourier-ratio, plots for the three cases of $\varepsilon = 2.0, 1.0, 0.5$, shown on chart 33, are repeated here together with their respective cardinal interpolants and zero-contour plots are shown here. The hexagonal patterns are clearly shown in the contour plots.
38	Expanded view of the cardinal interpolant for the hex-lattice 00-node. The cardinal interpolant $s_{Hex_{k\ell}}(x, y)$ for $k = \ell = 0$, that is for the $\overline{00}^{th}$ node, becomes $$s_{Hex_{00}}(x, y) = \frac{3}{4\pi^2}\frac{-2x\cos\frac{4\pi x}{3} + \left(x - \sqrt{3}y\right)\cos\left[\frac{2\pi}{3}\left(x - \sqrt{3}y\right)\right] + \left(x + \sqrt{3}y\right)\cos\left[\frac{2\pi}{3}\left(x + \sqrt{3}y\right)\right]}{x\left(x^2 - 3xy^2\right)}.$$ Using the Mathematica "Plot3D" function, we obtain this plot of the $\overline{00}^{th}$ cardinal interpolant for the doubly infinite uniform Hexagonal lattice. It rises to a value of one at $x = 0, y = 0$.
39	2-D RBF interpolant with 168 hexagonal stacked nodes. that collates the data is then determined by the double summation of the cardinal interpolants $s_{Hex_{k\ell}}(x, y)$ weighted by the double set of eigenvalues $\lambda_{k\ell}$ determined from the process described in Chart 15. The 12 x 14 Hexagonal-stacked nodes are shown on the right.
40	Max norm error w.r.t. test function for Cartesian and hexagonal lattices. Numerical experiments were made using the above formulations for the Cartesian and hexagonal lattice configurations from data bases generated from a known test function $f(x, y)$. The maximum norm error between $s_{Cart}(x, y)$ and $f(x, y)$ and in-turn between $s_{Hex}(x, y)$ and $f(x, y)$ are plotted here. The minimum errors are approximately $7.2 \cdot 10^{-4}$ and $3.8 \cdot 10^{-3}$ for the Hexagonal and Cartesian lattices, respectively—a factor of approximately five better for the hexagonal stacking. The asymptotic errors as $\varepsilon \to 0$ are $1.2 \cdot 10^{-3}$ and $5.2 \cdot 10^{-3}$ for the hexagonal and Cartesian lattices, respectively—again a factor of approximately five better for the more closely stacked hexagonal case.
41	Future effort—Extension to 3-D. Since most real-world problems involving partial differential equations involve three-dimensional Euclidian space, the extension of radial basis function techniques to 3-D is of paramount importance. Fortunately, Schoenberg's theorem assuring non-singularity of the system $\underline{\underline{A}} \cdot \underline{\lambda} = \underline{f}$ for radial functions that are infinitely differentiable and have positive definite Fourier transforms is applicable for any dimensionality. Thus , the 3-D case is, of course, covered. That is, referring to Chart 15, the eigenvalues, λ_k, are determinable, where each component RBF $\phi\left(\left\|\underline{x}_i - \underline{x}_j\right\|\right)$ is known for each known

	value of $\underline{x}_i = (x_i, y_i, z_i)$ and $\underline{x}_j = (x_j, y_j, z_j)$. In addition the triple Fourier transforms of the commonly used RBFs have been determined and are readily available. Therefore, the Poisson sums and Fourier ratios are readily extendable for 3-D Cartesian stacked lattices. However, the carry over from Cartesian to hexagonal stacking as shown by Charts 24 and 32 is not obvious in the 3-D case. Furthermore, there are two types of closely packed 3-D lattices that are superior to the 3-D Cartesian stacking. These are referred to as HCP (hexagonal closest packing) and FCC (face-centered cube) lattices, the latter also known as canon-ball stacking. Both appear in nature and therefore both may be necessary to study for RBF applications.